Journal of Statistical Physics, Vol. 94, Nos. 3/4, 1999

Some Properties of *k*-Step Exclusion Processes

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Received February 5, 1998; final October 8, 1998

We introduce k-step exclusion processes as generalizations of the simple exclusion process. We state their main equilibrium properties when the underlying stochastic matrix corresponds to a random walk or is positive recurrent and reversible. Finally, we prove laws of large numbers for tagged and second-class particles.

KEY WORDS: Interacting particle systems; invariant measures; tagged particle; second-class particle; exclusion process; k-step exclusion process.

1. INTRODUCTION

Let $\mathbf{S}$ be a finite or countable set, and $p(\cdot, \cdot)$ a transition matrix of a discrete Markov chain on $\mathbf{S}$.

For $k \in \mathbf{N} := \{1, 2, 3,...\}$ (the set of positive integer), a k-step exclusion process is a natural generalization of a simple exclusion process. It is a continuous time Markov process of state space

$\mathbf{X} = \{0, 1\}^{\mathbf{S}}$, the set of configurations of particles distributed on $\mathbf{S}$ with at most one particle per site.

Informally this process can be described in the following way: on each site of $\mathbf{S}$ we have one clock that rings (independently of others clocks) at random exponential times with parameter 1. If a particle is present at site x when the clock associated to this site rings, then the particle occupies the first vacant site encountered in the sequence $(X_n)_{1 \leqslant n \leqslant k}$, where $\{X_n\}_n$ is the Markov chain with probability transition $p(\cdot, .)$ starting at x (site x itself is consider vacant for the attempt, i.e., if the chain returns to x before encountering an empty site then the particle stays at x). If no empty site

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0022-4715/99/0200-0495$16.00/0

is encountered during these k attempts then the movement is cancelled and the particle stays at x (waiting for the next of its clock rings).

When $k=1$ the process is exactly the simple exclusion process introduced by Spitzer (1970).[19] This process has been intensively studied since then; we refer the reader to Liggett's book (1985)[17] Chapter VIII for an overview of its principal properties.

When k goes to infinity it is plausible that these processes will approach the Long Range Exclusion Process (see Guiol (1995)[8] or Andjel *et al.* (1998)[3]). In the Long Range Exclusion Process a particle that moves, follows the chain X_n until it finds an empty site, so that particles can travel long distance in short time. This process was also introduced by Spitzer (1970) in ref. 19 and studied systematically by Liggett (1980) in ref. 16 and in a lesser proportion by Zheng (1988)[20] and Guiol (1997).[9] A version of the k-step exclusion processes was used by Liggett in ref. 16 to approximate the long range exclusion process, but in that case particles disappear if they do not encounter a vacant site in their attempt to move.

Recent developments in the studies of long range processes such as the Long Range Exclusion Process,[9, 3] the Hammersley Process (see Aldous and Diaconis (1995)[1]), Self-organizing Particle Systems (see Carlson *et al.* (1993)[5]), the Toom model (see Lebowitz *et al.* (1996)[13]) show the existence of a real interest to have (Feller) approximations for these processes.

The aim of this paper is to give an overview of the principal properties of k-step exclusion processes.

In Section 2 we review some definitions and properties. In Section 3 are proved the following equilibrium results.

We denote by $\mathscr{I}_k$ the set of invariant measure of the k-step Exclusion Process.

Let $\rho: \mathbf{S} \to [0, 1]$, we denote by ν_ρ the product measure such that

$$\nu_\rho\{\eta(x_i)=1, 1 \leqslant i \leqslant n\} = \prod_{i=1}^{n} \rho(x_i)$$

for all $n \in \mathbf{Z}^+ := \{0, 1, 2,...\}$ (the set of nonnegative interger), $x_i \in \mathbf{S}$, $1 \leqslant i \leqslant n$.

Suppose that $p(\cdot, \cdot)$ is reversible. Then we have

Theorem 1.1. Let $p(\cdot, \cdot)$ be reversible with π i.e.,

$$\pi(x)\, p(x, y) = \pi(y)\, p(y, x) \qquad \text{for all} \quad x, y \in \mathbf{S}$$

and

$$\rho(x) = \frac{\pi(x)}{1+\pi(x)} \tag{1}$$

Then for all $k \geqslant 1$

$$\nu_\rho \in \mathscr{I}_k$$

This result is absolutely similar to the one in the simple exclusion case (see Liggett (1985) Chapter VIII, p. 380). We recall that for this last process the double stochasticity of the transition matrix $p(\cdot, \cdot)$ was also a sufficient condition to show that Bernoulli product measures with constant densities were invariant measures too. At the beginning of Section 3 we give a counter example for which this result is no more true for 2-step exclusion process.

Nevertheless, for a random walk on $\mathbf{Z}^d$ we have the (familiar) results:

Theorem 1.2. Suppose $p(x, y) = p(0, y-x)$ for all $x, y \in \mathbf{Z}^d$ then

$$\nu_\alpha \in \mathscr{I}_k$$

for every constant $\alpha \in [0, 1]$.

Furthermore in this base Bernoulli measures with constant densities are the only (ergodic) invariant measures that are translation invariant. We denote by $\mathscr{S}$ the set of translation invariant measures, $(\mathscr{S})_e$ the set of ergodic elements of $\mathscr{S}$ and $(\mathscr{I}_k)_e$ the set of extremal elements of $\mathscr{I}_k$.

Theorem 1.3. Suppose $p(x, y) = p(0, y-x)$ for all $x, y \in \mathbf{Z}^d$ and $p(\cdot, \cdot)$ irreducible then

$$(\mathscr{I}_k \cap \mathscr{S})_e = \{\nu_\alpha : \alpha \in [0, 1]\}$$

When the transition matrix $p(\cdot, \cdot)$ is reversible and positive recurrent (ergodic) we are able, as for the simple exclusion process, to characterize all the invariant measures. In this case there is a unique reversible probability stationary measure (for the chain $p(\cdot, \cdot)$), π, so that, according to Theorem 1.1, $\nu_\rho \in \mathscr{I}_k$, with ρ defined as in (1). Denote by $\nu^{(n)}$ the measure

$$\nu^{(n)}(\cdot) = \nu_\rho\left(\cdot \,\middle|\, \sum_x \eta(x) = n\right)$$

It is not difficult to prove that if $p(\cdot, \cdot)$ is irreducible, then these measures are invariant and extremal.

Theorem 1.4. Suppose that $p(x, y)$ is a positive recurrent, reversible, and irreducible matrix on $\mathbf{S}$. Then

(a) $(\mathscr{I}_k)_e = \{\nu^{(n)}, 0 \leqslant n \leqslant \infty\}$;

(b) If $\mu\{\eta: \sum_x \eta(x) = n\} = 1$, then $\lim_{t \to \infty} \mu S_k(t) = \nu^{(n)}$ for $0 \leqslant n \leqslant \infty$, where $\nu^\infty = \nu^{(\infty)}$ is the measure concentrated on $\eta \equiv 1$.

This result is also a generalization to the Instep exclusion of a simple exclusion result (see Liggett (1985),[17] p. 384).

In Section 4 we state some properties for a tagged particle and for a second class particle involved in a k-step exclusion processes. In that section we suppose $\mathbf{S} = \mathbf{Z}^d$ and $p(x, y) = p(0, y - x)$ for all $x, y \in \mathbf{Z}^d$ (i.e., $p(\cdot, \cdot)$ corresponds to a random walk on $\mathbf{Z}^d$).

Informally a tagged particle behaves exactly like a regular particle of the system. A second class particle moves like other particles (with the exclusion rule) except that when a regular ("first class") particle tries to jump onto it, they have to exchange positions.

We first concentrate on the k-step exclusion process as seen from a tagged particle. Tagged particles for the simple exclusion process were introduced by Spitzer (1970),[19] and were studied by Arratia (1983),[4] Ferrari (1986)[6] and Saada (1987).[18]

We generalize to k-step exclusion some results of invariance and ergodicity of product Bernoulli measures as seen from a tagged particle obtained from ref. 6. For the notation of the following results we refer the reader to Section 4.1.

Theorem 1.5. (a) The Palm measure $\hat{\nu}_\rho$ of ν_ρ is invariant for the tagged particle process, i.e.,

$$\hat{\nu}_\rho \hat{S}_k(t) = \hat{\nu}_\rho$$

(b) Furthermore if $p(\cdot, \cdot)$ is irreducible

$$(\hat{\mathscr{I}}_k \cap \mathscr{S})_e = \{\hat{\nu}_\rho : 0 \leqslant \rho \leqslant 1\}$$

The second class particle was introduced to analyze the shock structure and its fluctuations for the simple exclusion process (see Ferrari and Fontes (1993)[7] for a review on the topic). We show that all product Bernoulli measures with constant densities $p \in]0, 1]$ as seen from a second class particle are invariant if and only if $p(\cdot, \cdot)$ is symmetric:

Theorem 1.6. $\hat{\nu}_\rho \in \mathscr{I}_k$ for all $\rho \in]0, 1]$ if and only if $p(\cdot, \cdot)$ is symmetric.

In Section 5 we prove laws of large numbers for the tagged particle and for the second class particle, showing that X_t/t (where X_t is the position of the tagged particle at time t) and Y_t/t (where Y_t is the position of the second class particle at time t) converge a.s. and in L_1 (Theorems 5.1 and 5.4).

Although there are strong links and similarities between simple and k-step exclusion they do not always behave in the same way. For the symmetric case duality relations fail for the k-step exclusion processes. This technique was the key tool for the study of the symmetric simple exclusion process (see ref. 17, Chapter VIII, Section 1). Furthermore, results obtained in the symmetric case have also some relevance for asymmetric case. For instance the method to prove that Bernoulli product measures with constant densities are extremal for the asymmetric simple exclusion process (see the end of the proof of Proposition 2, p. 377 in Saada (1987)[18]) relies highly on the result for the symmetric case. In particular this last fact was one ingredient to prove the strong laws of large numbers for the tagged particle for the simple exclusion case in Saada (1987).[18]

2. DEFINITIONS AND REMARKS

Let $\mathbf{S}$ be a finite or countable set, $\{X_n\}_{n \in \mathbf{Z}^+}$ be a Markov chain on $\mathbf{S}$ with transition matrix $p(\cdot, \cdot)$, $\mathbf{P}^x(X_0 = x) = 1$. Let $k \in \mathbf{N}$, under the mild hypothesis

$$\sup_{y \in \mathbf{S}} \sum_{x \in \mathbf{S}} p(x, y) < +\infty$$

one can define a continuous semi-group $S_k(t)$ on $C(\mathbf{X})$ (see ref. 17, p. 30) with infinitesimal generator Ω_k given by: for all cylinder function f,

$$\Omega_k f(\eta) = \sum_{\eta(x) = 1,\, \eta(y) = 0} q_k(x, y, \eta)[f(\eta_{xy}) - f(\eta)] \tag{2}$$

where $q_k(x, y, \eta) = \mathbf{E}^x[\prod_{i=1}^{\sigma_y - 1} \eta(X_i), \sigma_y \leqslant \sigma_x, \sigma_y \leqslant k]$ is the intensity for moving from x to y on configuration η and $\sigma_y = \inf\{n \geqslant 1 \colon X_n = y\}$ is the first (non zero) arrival time at site y of the chain starting at site x.

By Hille-Yosida's theorem the closure of Ω_k corresponds to a continuous Markov semi-group, which corresponds to the k-step exclusion process.

Remark 2.1. It if also possible to construct this process via the graphical construction due to Harris (1978).[11]

Let $\mathscr{P}$ be the set of probability measures on $\mathbf{X}$. When $\mathbf{S} = \mathbf{Z}^d$ we will denote by $\mathscr{S}$ the set of elements of $\mathscr{P}$ which are translation invariant.

For every $\mu \in \mathscr{P}$ we denote by $\mu S_k(t)$ the probability measure defined by

$$\int f \, d\mu \, S_k(t) = \int S_k(t) \, f \, d\mu$$

for all $f \in C(\mathbf{X})$.

Let $\mathscr{D}$ be the set of cylinder functions on $\mathbf{X}$, and let $\mathscr{I}_k$ denote the set of elements of $\mathscr{P}$ which are invariant by the k-step exclusion process, i.e.,

$$\begin{aligned} \mathscr{I}_k &= \{\mu \in \mathscr{P} : \mu S_k(t) = \mu \text{ for all } t\} \\ &= \left\{\mu \in \mathscr{P} : \int \Omega_k f \, d\mu = 0 \text{ for all } f \in \mathscr{D}\right\} \end{aligned} \tag{3}$$

Remark 2.2. For all $k \in \mathbf{N}$, k-step exclusion process has the Feller property:

$$\text{for all} \quad f \in C(\mathbf{X}) \qquad \text{and all} \quad t > 0, \qquad \text{we have} \quad S_k(t) \, f \in C(\mathbf{X})$$

The second equality in (3) is due to that fact.

Remark 2.3. One can write the rates in the following way: for all $k \in \mathbf{N}$, $x, y \in \mathbf{S}$ and $\eta \in \mathbf{X}$ we have

$$\begin{aligned} q_k(x, y, \eta) = p(x, y) + \sum_{z \neq x, y} p(x, z) \, p(z, y) \, \eta(z) + \cdots \\ + \sum_{z_1 \neq x, y, \ldots, z_{k-1} \neq x, y} p(x, z_1) \cdots p(z_{k-1}, y) \, \eta(z_1) \cdots \eta(z_{k-1}) \end{aligned}$$

In particular we have for any k, x, y, η

$$q_k(x, y, \eta) \geqslant p(x, y)$$

Remark 2.4. There is a lack of monotonicity in k of these processes. i.e., if $f \in C(\mathbf{X})$ is a monotone increasing function, it does not imply $S_{k+1}(t) \, f \geqslant S_k(t) \, f$.

This can be seen in the following simple example.

Let $\mathbf{S} = \{0, 1, 2\}$, $p(0, 1) = p(1, 2) = p(2, 0) = 1$. Denote by η the initial configuration such that $\eta(0) = \eta(1) = 1$, $\eta(2) = 0$. Then construct a coupling $(\xi_t^\eta, \zeta_t^\eta)$ with ξ_t^η and ζ_t^η being respectively the 2-step exclusion process and the simple exclusion process starting from configuration η. Then $\xi_0^\eta = \zeta_0^\eta = \eta$, but with a positive probability the particles on site 0 will try to jump before the particles on site 1, if this happens at time δ^- then $\xi_\delta^\eta \nleqslant \zeta_\delta^\eta$ and $\xi_\delta^\eta \ngeqslant \zeta_\delta^\eta$. ∎

3. EQUILIBRIUM RESULTS

Remark 3.1. The correspondence between the simple exclusion process and the k-step exclusion processes is not as easy as expected. For instance the condition $p(\cdot,\cdot)$ doubly stochastic does not assure anymore the invariance of the product measure with constant densities when $k \geqslant 2$, as one can see in the following example.

Let $\mathbf{S} = \{-1, 0, 1\}$ suppose that $p(-1,0) = p(-1,1) = p(0,1) = 1/2$, $p(0,-1) = p(1,0) = 1/3$, $p(-1,-1) = p(1,1) = 0$, $p(0,0) = 1/6$, $p(1,-1) = 2/3$. Then $p(\cdot,\cdot)$ is doubly stochastic so that the product measure ν_α is invariant for the simple exclusion process for every constant $\alpha \in [0,1]$ (cf. Theorem 2.1 (a) Chap. VIII in ref. 17). But a simple calculation using the generator of the 2-step exclusion process gives

$$\int \Omega_2 \mathbf{1}\{\eta \in \mathbf{X}: \eta(-1) = \eta(0) = 1\}\, d\nu_\alpha = \frac{\alpha^2(1-\alpha)}{36} \neq 0$$

for every constant $\alpha \in]0,1[$. This means that these measures are not invariant for the 2-step exclusion process for that particular $p(\cdot,\cdot)$.

Nevertheless, to characterize all the invariant and translation invariant measures is most of the time an easy generalization of the simple exclusion case.

The following properties are some generalizations of well known results for the simple exclusion process.

3.1. Proofs of Theorem 1.1 and 1.2

We will prove the following proposition which contains both results.

Proposition 3.2. For all $k \geqslant 1$,

(a) Suppose there exists $\pi(\cdot): \mathbf{S} \to \mathbf{R}^+$ such that

$$\pi(x)\, p(x,y) = \pi(y)\, p(y,x) \qquad \text{for all} \quad x, y \in \mathbf{S} \tag{4}$$

Then,

$$\nu_\rho \in \mathscr{I}_k$$

where $\rho(x) = \pi(x)/[1+\pi(x)]$.

(b) If $\{p(\cdot,\cdot)\}$ is translation invariant then for all constant $\alpha \in [0,1]$, we have

$$\nu_\alpha \in \mathscr{I}_k$$

Proof. Part (a) is a straight-forward adaptation of part (b) of Theorem 2.1, p. 380 in Liggett's book.[17] For a detailed proof we refer the reader to ref. 8.

Part (b) follows the same ideas but need more care. We prove (b) for $k=2$. The general case consists in gathering terms that annihilate each other. As it is quite long and tedious no more detail will be given.

For all $A \subset \mathbf{Z}^d$, $|A| = n$ finite and $f_A(\eta) = \prod_{u \in A} \eta(u)$ we need to show that

$$\int \Omega_k f_A \, d\nu_\alpha = 0$$

Applying (2) to f_A:

$$\int \Omega_k f_A(\eta) \, d\nu_\alpha = \sum_{y \in A} \sum_{x \in A^c} \int q_k(x, y, \eta) \, \eta(x)[1-\eta(y)] \prod_{u \in A,\, u \neq y} \eta(u) \, d\nu_\alpha$$
$$- \sum_{y \in A} \sum_{x \in A^c} \int q_k(y, x, \eta) \, f_A(\eta)[1-\eta(x)] \, d\nu_\alpha \tag{5}$$

Using Remark 2.3 we obtain the sum of the three following terms (multiplied by $\alpha^n(1-\alpha)$)

$$\left\{ \sum_{y \in A} \sum_{x \in A^c} p(x, y) - \sum_{y \in A^c} \sum_{x \in A} p(x, y) \right\} \tag{6}$$

$$\left\{ \sum_{y \in A} \sum_{z \in A \setminus \{y\}} \sum_{x \in A^c} p(x, z) \, p(z, y) - \sum_{x \in A} \sum_{z \in A \setminus \{x\}} \sum_{y \in A^c} p(x, z) \, p(z, y) \right\} \tag{7}$$

$$\alpha \left\{ \sum_{x \in A^c} \sum_{z \in A^c \setminus \{x\}} \sum_{y \in A} p(x, z) \, p(z, y) - \sum_{y \in A^c} \sum_{z \in A^c \setminus \{y\}} \sum_{x \in A} p(x, z) \, p(z, y) \right\} \tag{8}$$

Those three terms are zero: for (6) we only have to add and subtract the same quantity in the brackets $\sum_{y \in A} \sum_{x \in A} p(x, y)$

$$\sum_{y \in A} \sum_{x \in \mathbf{S}} p(x, y) = |A| = \sum_{y \in \mathbf{S}} \sum_{x \in A} p(x, y)$$

For (7) we add and subtract $\sum_{y \in A} \sum_{z \in A} \sum_{x \in A} p(x, z)\, p(z, y)$, and since

$$\sum_{y \in A} \sum_{z \in A} \sum_{x \in \mathbf{S}} p(x, z)\, p(z, y) = \sum_{y \in A} \sum_{z \in A} p(z, y)$$
$$= \sum_{z \in A} \sum_{x \in A} p(x, z)$$
$$= \sum_{y \in \mathbf{S}} \sum_{z \in A} \sum_{x \in A} p(x, z)\, p(z, y)$$

denoting $p = p(x, x)$ we have

$$p \left\{ \sum_{y \in A} \sum_{x \in A^c} p(x, y) - \sum_{y \in A^c} \sum_{x \in A} p(x, y) \right\}$$

which is null by the preceding argument.

Finally for (8) we add and subtract $\sum_{y \in A} \sum_{z \in A^c} \sum_{x \in A} p(x, z)\, p(z, y)$ and since

$$\sum_{y \in A} \sum_{z \in A^c} \sum_{x \in \mathbf{S}} p(x, z)\, p(z, y)$$
$$= \sum_{y \in A} \sum_{z \in A^c} p(z, y) = \sum_{z \in A^c} \sum_{x \in A} p(x, z) = \sum_{y \in \mathbf{S}} \sum_{z \in A^c} \sum_{x \in A} p(x, z)\, p(z, y)$$

it remains

$$p \left\{ \sum_{y \in A^c} \sum_{x \in A} p(x, y) - \sum_{y \in A} \sum_{x \in A^c} p(x, y) \right\}$$

which is null as in the first step. This concludes (b). ∎

3.2. Ergodicity

We will only sketch the proof of Theorem 1.3 as it is very standard (see Liggett's book Chapter VIII, for instance).

As these processes are Feller it is possible to show that if μ_1 and μ_2 are in $(\mathscr{I}_k \cap \mathscr{S})_e$ then one can find a coupled measure $\tilde{v} \in (\tilde{\mathscr{I}}_k \cap \mathscr{S})_e$ with marginals μ_1 and μ_2. Using basic coupling then one has to show that if $\tilde{v} \in (\tilde{\mathscr{I}}_k \cap \mathscr{S})$ then $\tilde{v}\{\eta \geqslant \xi \text{ or } \eta \leqslant \xi\} = 1$. Those two points are sufficient to conclude (see Andjel (1981)[2] for instance). (A complete detailed proof can be found in Guiol (1995)[8]).

3.3. Proof of Theorem 1.4

(a) is a consequence of (b), and (b) is a straight-forward adaptation of the proof of Theorem 2.17, p. 384 of Liggett's book.[17] For a very detailed proof we refer the reader to ref. 8. ▌

4. TAGGED AND SECOND CLASS PARTICLES IN THE *k*-STEP EXCLUSION PROCESS

We are interested in the k-step exclusion process as seen from a tagged particle (respectively a second class particle).

For all these processes the origin will always be occupied.

For the tagged particle, after a random exponential time with mean 1, a transition of the system will occur moving the origin to X_r, where $\{X_n\}$ is a Markov chain with probability transition $p(\cdot, \cdot)$, $X_0 = 0$ and $\tau = \inf\{1 \leqslant n \leqslant k: \eta(X_n) = 0\}$. When $\{1 \leqslant n \leqslant k : \eta(X_n) = 0\} = \varnothing$ we set $X_\infty = 0$ and the system does not move. The other particles move as in the k-step exclusion process.

For the second class particle we add to the preceding rules that after a random exponential time mean 1, the system is translated putting the origin on y if $\eta(y) = 1$ with probability $q_k(y, 0, \eta)$. So the second particle moves like a standard particle but when another particle arrives on its site (here the origin) then the two particles exchange their positions.

4.1. Notations and Definitions

We will suppose $\mathbf{S} = \mathbf{Z}^d$ and $p(x, y) = p(0, y - x)$.

The state space will be a little modified:

$$\hat{\mathbf{X}} = \mathbf{X} \cap \{\eta: \eta(0) = 1\}$$

The k-step exclusion process as seen from a tagged particle has for state space $\hat{\mathbf{X}}$ and for generator:

$$\hat{\Omega}_k f(\eta) = \sum_{\eta(x) = 1,\, \eta(y) = 0 x,\, y \neq 0} q_k(x, y, \eta)[f(\eta_{xy}) - f(\eta)]$$

$$+ \sum_{\eta(y) = 0,\, y \neq 0} q_k(0, y, \eta)[f(\tau_{-y}\eta_{0y}) - f(\eta)]$$

where $\tau_{-y}\eta(z) = \eta(z + y)$.

We will also consider the k-step exclusion process as seen from a second class particle on $\hat{\mathbf{X}}$ with generator:

$$\bar{\Omega}_k f(\eta) = \hat{\Omega}_k f(\eta) + \breve{\Omega}_k f(\eta)$$

where

$$\breve{\Omega}_k f(\eta) = \sum_{\eta(x)=1,\, x \neq 0} q_k(x, 0, \eta)[f(\tau_{-x}\eta_{0x}) - f(\eta)]$$

By Liggett's criterion:[14]

$$\sum_{\eta(x)=1,\, \eta(y)=0x,\, y \neq 0} q_k(x, y, \eta)[f(\eta_{xy}) - f(\eta)]$$

is a generator of a contractive semi-group and

$$\sum_{\eta(y)=0,\, y \neq 0} q_k(0, y, \eta)[f(\tau_{-y}\eta_{0y}) - f(\eta)]$$

(respectively

$$\sum_{\eta(x)=1,\, x \neq 0} q_k(x, 0, \eta)[f(\tau_{-x}\eta_{0x}) - f(\eta)])$$

is a bounded operator. Hence by Theorem 2 of Gustafson[10] $\hat{\Omega}_k$ and $\bar{\Omega}_k$ are infinitesimal generators of contractive semi-groups that we denote respectively by $\hat{S}(t)$ and $\bar{S}(t)$. To each of these semi-groups corresponds a unique Markov process on $\hat{\mathbf{X}}$ with respective generators $\hat{\Omega}_k$ and $\bar{\Omega}_k$.

Let $\mathscr{P}(\hat{\mathbf{X}})$ be the set of probability measures on $\hat{\mathbf{X}}$, and $\hat{\mathscr{I}}_k = \{\mu \in \mathscr{P}(\hat{\mathbf{X}}): \mu \hat{S}_k(t) = \mu\}$ (respectively $\mathscr{I}_k = \{\mu \in \mathscr{P}(\hat{\mathbf{X}}): \mu \bar{S}_k(t) = \mu\}$) the set of invariant measures for the k-step exclusion process as seen from the tagged particle (respectively the second class particle).

4.2. Invariant Measures

Let $\mu \in \mathscr{S}$ be such that $\beta(\mu) = \mu\{\eta(0) = 1\} > 0$. We define the Palm measure of μ, by $\hat{\mu}$, a measure on $\hat{\mathbf{X}}$ such that

$$\hat{\mu} = \mu(\cdot \mid \eta(0) = 1)$$

(Let us recall that if μ puts all its mass on the empty configuration $\eta \equiv 0$ (i.e., $\mu = \delta_0$) one defines $\hat{\mu} = \delta_{\{0\}}$ as the measure that puts all its mass on configuration with only one particle at 0).

We denote

$$\hat{\mathscr{S}} = \{\hat{\mu}: \mu \in \mathscr{S}\}$$

Proof of Theorem 1.5. The proof is a straightforward adaptation of the proofs of theorems 2.3 and 2.4 of Ferrari in ref. 6. ∎

Proof of Theorem 1.6. For this we prove the following proposition that contains the theorem.

Proposition 4.1. For all $k \geqslant 1$,

(a) If $p(\cdot, \cdot)$ is symmetric and $\mu \in \mathscr{S}$ then $\hat{\mu} \in \bar{\mathscr{I}}_k$.

(b) Let $\rho \in]0, 1]$, $\hat{\nu}_\rho \in \bar{\mathscr{I}}_1$ if and only if $p(\cdot, \cdot)$ is symmetric.

(c) Let $\rho \in]0, 1]$, for $\mathbf{S} = \mathbf{Z}$ if $p(x, x+1) = 1 - p(x, x-1) = p$ then $\hat{\nu}_\rho \in \bar{\mathscr{I}}_k$ if and only if $p(\cdot, \cdot)$ is symmetric (i.e., $p = 1/2$).

(d) Except (possibly) for a finite number ($\leqslant k$) of densities we have

$$\hat{\nu}_\rho \in \bar{\mathscr{I}}_k \text{ implies } p(\cdot, \cdot) \text{ is symmetric}$$

Remark 4.2. (a) and (d) prove Theorem 1.6.
Unfortunately we are not able to prove in a general setting that

$$\hat{\nu}_\rho \in \bar{\mathscr{I}}_k \qquad \text{if and only if} \quad p(\cdot, \cdot) \text{ is symmetric}$$

for every density $\rho > 0$.

Items (b) and (c) show that this is true for the simple exclusion and for the k-step exclusion with nearest neighbor jumps in dimension 1 respectively.

Proof. Note that if $\mu \in \mathscr{S}$ then

$$\int T\bar{\Omega}_k f(\eta)\, d\mu = \int \Omega_k Tf(\eta)\, d\mu + \int T\breve{\Omega}_k f(\eta)\, d\mu \tag{9}$$

where

$$\breve{\Omega}_k f(\eta) = \sum_{\eta(x) = 1 x \neq 0} q_k(x, 0, \eta)[f(\tau_{-x}\eta_{0x}) - f(\eta)]$$

We try to find conditions for which for all $f \in \mathscr{D}$

$$\int T\breve{\Omega}_k f(\eta)\, d\mu = 0$$

$$\int T\breve{\Omega}_k f(\eta)\, d\mu = \sum_{x \neq 0} \int q_k(x, 0, \eta)\, \eta(0)\, \eta(x) [f(\tau_{-x}\eta_{0x}) - f(\eta)]\, d\mu \tag{10}$$

and since $\eta_{0x} = \eta$ when $\eta(x) = \eta(0)$, by translation invariance of μ we obtain

$$= \sum_{x \neq 0} \left[\int q_k(0, -x, \eta)\, \eta(-x)\, \eta(0)\, f(\eta)\, d\mu - \int q_k(x, 0, \eta)\, \eta(0)\, \eta(x)\, f(\eta)\, d\mu \right]$$

$$= \sum_{x \neq 0} \int [q_k(0, x, \eta) - q_k(x, 0, \eta)]\, \eta(0)\, \eta(x)\, f(x)\, d\mu \tag{11}$$

which is zero if $p(\cdot, \cdot)$ is symmetric and shows part (a).

Now let $\mu = \nu_\rho$, for $\rho \in]0, 1[$. For all $y \in \mathbf{Z}^d \backslash \{0\}$ let $f_y = \eta(y)$. Then writing (11) for f_y and developing coefficients $q_k(.,.,.)$ we obtain the three following cases:

— **1st case** if $k = 1$

$$\int T\breve{\Omega}_k f_y(\eta)\, d\mu = \rho^2 \left([p(0, y) - p(y, 0)] + \rho \sum_{x \neq 0 x \neq y} [p(0, x) - o(x, 0)] \right)$$

and as $p(\cdot, \cdot)$ is doubly stochastic

$$= \rho^2 (1 - \rho)[p(0, y) - p(y, 0)]$$

This last expression is null for all y only if $p(\cdot, \cdot)$ is symmetric. This ends part (b);

— **2nd case** if $d = 1$ ($\mathbf{S} = \mathbf{Z}$) and $p(x, x+1) = 1 - p(x, x-1) = p$, $q = 1 - p$; without loss of generality we take y such that $|y| > k$, then (11) is a polynomial in ρ degree $k+2$ whose order $2l+1$ coefficients, $3 \leqslant 2l+1 \leqslant k+2$ are

$$(p^{2l+1} - q^{2l+1}) + C_{2l+1}^{2l} pq(p^{2l-1} - q^{2l-1}) + \cdot s + C_{2l+1}^{l+1} p^l q^l (p - q)$$

and order $2l$, $3 \leqslant 2l \leqslant k+2$ are

$$(p^{2l} - q^{2l}) + C_{2l}^{2l-1} pq(p^{2l-2} - q^{2l-2}) + \cdots + C_{2l}^{l+2} p^l q^l (p^2 - q^2)$$

Since these coefficients have the same sign then (11) is null only if $p = q = 1/2$. This ends part (c);

— **3rd case** For part (d) without loss of generality we may suppose that $p(x, x) = 0$, we then obtain a polynomial in ρ of degree $k+2$ for which 0 is a root of order two and so it has at most k non null roots. Its lowest degree term is

$$\rho^2 [p(0, y) - p(y, 0)]$$

This term is constantly null only if $p(\cdot, \cdot)$ is symmetric. ∎

5. LAW OF LARGE NUMBERS

5.1. Introduction

We still suppose that $p(\cdot, \cdot)$ is translation invariant and $\mathbf{S} = \mathbf{Z}^d$.

We are interested in the asymptotic behavior of a tagged particle (respectively a second class particle) originally at $x \in \mathbf{S}$, when the other particles of the system are distributed according to ν_ρ, the Bernoulli product measure ($0 \leqslant \rho \leqslant 1$).

For $k = 1$, supposing that $p(\cdot, \cdot)$ has a finite first moment, F. Spitzer(19) has computed $\mathbf{E} X_t$, where X_t is the position of the tagged particle at time t, and has shown the existence of an almost sure limit for X_t / t. C. Kipnis(12) (in the nearest neighbors case in dimension one) and E. Saada(18) in the other cases have shown that this limit is constant and equal to

$$\lim_{t \to \infty} \frac{X_t}{t} = \sum_{y \in \mathbf{Z}^d} y p(0, y) \quad \text{a.s.}$$

5.2. Tagged Particle

For the tagged particle in the k-step exclusion process we have

Theorem 5.1. Suppose $p(\cdot, \cdot)$ has a bounded first moment. If the k-step exclusion process as seen from the tagged particle has for initial distribution $\hat{\nu}_\rho$ then,

$$\mathbf{E}X_t = x + (1-\rho)\, t \sum_y yp(0, y) + \cdots$$
$$+ (1-\rho)\, t \sum_y \sum_{z_1,\ldots, z_{k-1} \neq 0,\, y} \rho^{\gamma} yp(0, z_1) \cdots p(z_{k-1}, y)$$
$$\text{where} \quad \gamma(z_1,\ldots, z_{k-1}) = \mathrm{Card}\{z_1,\ldots, z_{k-1}\}$$

and

$$\lim_{t \to \infty} \frac{X_t}{t} \quad \text{exists a.s. and in } L^1$$

Proof. It's a corollary of Theorem 1.5 (see Liggett,[17] p. 396). By this theorem X_t is stationary increasing. Then $\mathbf{E}X_t$ is a linear function of t.

On the other hand

$$\lim_{t \downarrow 0} \frac{\mathbf{E}X_t - x}{t} = (1-\rho) \sum_y yp(0, y) + \cdots$$
$$+ (1-\rho) \sum_y \sum_{z_1,\ldots, z_{k-1}} \rho^{\gamma(z_1,\ldots, z_{k-1})} yp(0, z_1) \cdots p(z_{k-1}, y)$$

we have the first result.

The a.s. convergence of X_n/n is a consequence of the Ergodic theorem. Noting that

$$\mathbf{E} \sup_{0 \leqslant t \leqslant 1} \|X_t\| < \infty$$

and using a Borel-Cantelli argument we deduce the convergence for all t. ∎

We then state a natural conjecture:

Conjecture 5.2. For $k \geqslant 2$, under the hypothesis of Theorem 5.1

$$\lim_{t \to \infty} \frac{X_t}{t} = (1-\rho) \sum_y yp(0, y) + \cdots$$
$$+ (1-\rho) \sum_y \sum_{z_1,\ldots, z_{k-1}} \rho^{\gamma(z_1,\ldots, z_{k-1})} yp(0, z_1) \cdots p(z_{k-1}, y) \quad \text{a.s.}$$

Remark 5.3. The major difficulty to prove this is to shots that

$$\nu_\rho \in (\mathscr{I}_k)_e$$

when $k \geqslant 2$.

Using coupling arguments of Pigged (1976) in ref. 15 it is possible, at least for the nearest neighbor asymmetric case (i.e., $d=1$, $p(x, x+1)=p$, $p(x, x-1)=1-p$, $p \neq 1/2$), to show that $\nu_\rho \in (\mathcal{I}_k)_e$.

5.3. Second-Class Particle

Let Y_t be the position at time t of the second class particle originally at $x \in \mathbf{Z}^d$.

Theorem 5.4. With the hypothesis of Theorem 5.1 and if $p(\cdot, \cdot)$ is symmetric then

$$\mathbf{E} Y_t = x \qquad \text{and} \qquad \lim_{t \to \infty} \frac{Y_t}{t} \text{ exists a.s. and in } L^1$$

Proof. The proof parallels the one for the tagged particle. ∎

ACKNOWLEDGMENTS

This research is part of FAPESP's Projeto Temático 95/0790-1 and Pronex No 41.96.0923.00, and was also supported by FAPESP Grant 96/04859-9. Part of this paper is an adaptation of the author's Ph.D. thesis at the CMI de l'Université de Provence, Marseille. Thanks are given to my adviser, Enrique Andjel, and to Ellen Saada, Dominique Bakry, Pablo Ferrari and Mihail Menshikov for valuable discussions. Special thanks to Krisnamurthi Ravishankar and Ellen Saada for encouraging me to publish this paper, and to the Referees for their careful reading and their help to make this paper clearer. Final thanks for the kind hospitality of the Department of Statistics at IME-USP in São Paulo where this work has been done.

REFERENCES

1. D. Aldous and P. Diaconis, Hammersley's interacting particle process and longest increasing subsequences, *Probab. Theory Related Fields* **103**(2):199–213 (1995).
2. E. D. Andjel, The asymmetric simple exclusion process on $\mathbf{Z}^d$, *Z. Wahrsch. Verw. Gebiete* **58**:423–432 (1981).
3. E. D. Andjel and H. Guiol, In Preparation.
4. R. Arratia, The motion of a tagged particle in the simple symmetric exclusion on $\mathbf{Z}$, *Ann. Probab.* **11**:362–373 (1983).
5. J. M. Carlson, E. R. Grannan, G. H. Swindle, and J. Tour, Singular diffusion limits of a class of reversible self-organizing particle systems, *Ann. Probab.* **21**(3):1372–1393 (1993).
6. P. A. Ferrari, The simple exclusion process as seen from a Tagged Particle, *Ann. Probab.* **14**:1277–1290 (1986).

7. P. A. Ferrari and L. R. G. Fontes, Shocks in asymmetric one-dimensional exclusion processes, *Resenhas* **1**(1):57–68 (1993).
8. H. Guiol, Thèse de doctorat de l'Université de Provence, France, 1995.
9. H. Guiol, Un résultat pour le processus d'exclusion à longue portée, *Ann. Inst. Henri Poincaré*, Vol 33, No 4, (1997), 387-405.
10. K. Gustafson, A perturbation lemma, *Bull. Amer. Math. Soc.* **72**:334–338 (1966).
11. T. E. Harris, Additive set-valued Markov processes and graphical methods, *Ann. Probab.* **6**:355–378 (1978).
12. C. Kipnis, Central limit theorems for infinite series of queues and applications to simple exclusion, *Ann. Probab.* **14**:397–408 (1986).
13. J. L. Lebowitz, C. Neuhauser, and K. Ravishankar, Dynamics of a spin-exchange model, *Stochastic Process. Appl.* **64**(2):187–208 (1996).
14. T. M. Liggett, Existence theorems for infinite particle systems, *Trans. Amer. Math. Soc.* **165**:471–481 (1972).
15. T. M. Liggett, Coupling the simple exclusion process, *Ann. Probab.* **4**:339–356 (1976).
16. T. M. Liggett, Long-range exclusion processes, *Ann. Probab.* **8**:861–889 (1980).
17. T. M. Liggett, *Interacting Particle Systems* (Springer, New York, 1985).
18. E. Saada, A limit theorem for the position of a tagged particle in a simple exclusion process, *Ann. Probab.* **15**:375–381 (1987).
19. F. Spitzer, Interaction of Markov processes, *Adv. Math.* **5**:247–290 (1970).
20. X. Zheng, Ergodic theorem for generalized long-range exclusion processes with positive recurrent transition probabilities, *Acta Math. Sinica* **4**:193–209 (1988).